2
Quantum theory of light

2.1 The electromagnetic oscillator

Light shows both wave and particle aspects. It propagates in space and interferes with itself, it disperses in optical media such as prisms, and it displays polarization effects. All these properties are commonly regarded as wave features. On the other hand, when detected with sufficiently high precision, light appears as distinct detector clicks called photons. We may say as well that light behaves like moving particles that nevertheless follow the rules of wave interference. This strange picture has puzzled countless people during much of this century. Strictly speaking, the picture has not been explained yet, but rather it has been formulated more precisely in the quantum theory of light. According to this theory, the wave features of light are regarded as classical aspects (which does not necessarily mean that the particle aspects are entirely quantum). This book focuses on the quantum aspects of light. We will use the most primitive concept for the classical wave features but a sophisticated machinery for the quantum aspects. Our model is the electromagnetic oscillator. One complex vector function $u(x, t)$ called a *spatial–temporal mode* comprises all classical wave aspects including polarization. The simplest example of a spatial–temporal mode is a plane wave

$$u(x, t) = u_0 \exp[\mathrm{i}(kx - \omega t)] \tag{2.1}$$

of polarization vector u_0, frequency ω, and wave vector k with $k^2 = \omega^2/c^2$. (As usual, c denotes the speed of light.) This mode defines a framework in space and time that may be excited by the quantum field "light." The mode function quantifies the strength of one excitation in space and time. Of course, the possibilities for setting the frame $u(x, t)$ are infinite as long as the spatial–temporal mode obeys the laws of classical waves, that is, Maxwell's equations. The choice of $u(x, t)$ is made by the observer. (We will study in Section 4.2.3 how this is accomplished in a particular type of experiment.) The observer singles out one mode, one quantum object, from the rest of the world. This object turns out to be a harmonic oscillator described by the annihilation operator $\hat{a}$. The operator $\hat{a}$ stands for the quantized amplitude with which the spatial–temporal mode can be excited. In classical optics it would be just a complex number α of magnitude $|\alpha|$ and phase $\arg\alpha$. The quantized amplitude $\hat{a}$ is neither predetermined nor given by the observer but depends on the state of the spatial–temporal mode. This state exists even if literally nothing is in the mode chosen by the observer. Then the light is just in the *vacuum state*. We will see later in this book that this "nothing" can indeed cause significant physical effects.

To make all these woolly words more precise and to cut a long story short, we postulate that the electric field strength $\hat{E}$ of the light field is given by

$$\hat{E} = u^*(x,t)\hat{a} + u(x,t)\hat{a}^\dagger \tag{2.2}$$

and that the amplitude operator $\hat{a}$ is a bosonic annihilation operator, that is, $\hat{a}$ obeys the commutation relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{2.3}$$

Throughout this book we set Planck's constant

$$\hbar = 1 \tag{2.4}$$

for simplicity. (This can be always achieved by a proper rescaling of physical units.)

In the following we introduce the key elements of quantum-oscillator physics. The *photon-number* operator $\hat{n}$ accounts for the photons in the chosen spatial–temporal mode and is given by the counterpart of a classical modulus-squared amplitude

$$\hat{n} \equiv \hat{a}^\dagger\hat{a}. \tag{2.5}$$

We introduce the *phase-shifting* operator

$$\hat{U}(\theta) \equiv \exp(-\mathrm{i}\theta\hat{n}). \tag{2.6}$$

As the name suggests, the phase-shifting operator provides the amplitude $\hat{a}$ with a phase shift θ when acting on $\hat{a}$

$$\hat{U}^\dagger(\theta)\hat{a}\hat{U}(\theta) = \hat{a}\exp(-\mathrm{i}\theta). \tag{2.7}$$

This property is easily seen by calculating the derivative of $\hat{U}^\dagger\hat{a}\hat{U}$ with respect to θ

$$\begin{aligned}\frac{d}{d\theta}\hat{U}^\dagger\hat{a}\hat{U} &= \mathrm{i}\left[\hat{n}, \hat{U}^\dagger\hat{a}\hat{U}\right] = \hat{U}^\dagger\mathrm{i}[\hat{n},\hat{a}]\hat{U} \\ &= -\mathrm{i}\hat{U}^\dagger\left[\hat{a},\hat{a}^\dagger\right]\hat{a}\hat{U} = -\mathrm{i}\hat{U}^\dagger\hat{a}\hat{U}.\end{aligned} \tag{2.8}$$

Because the right-hand side of Eq. (2.7) obeys the same differential equation with the initial operator $\hat{a}$ for $\theta = 0$, both sides must be equal indeed. There is another way of looking at formula (2.7). When the observer wishes to change the phase of the spatial–temporal mode, that is,

$$\hat{E} = u^*(x,t)\exp(-\mathrm{i}\theta)\hat{a} + u(x,t)\exp(+\mathrm{i}\theta)\hat{a}^\dagger, \tag{2.9}$$

this phase is picked up by the quantum amplitude $\hat{a}$. We may understand this to mean that the field state $\hat{\rho}$ has been altered by the observer to produce a new state

$$\hat{\rho}(\theta) = \hat{U}\hat{\rho}\hat{U}^\dagger \tag{2.10}$$

because any predictable quantity or expectation value depending on $\hat{a}\exp(-\mathrm{i}\theta) = \hat{U}^\dagger\hat{a}\hat{U}$ is reproduced when $\hat{\rho}$ is replaced by $\hat{\rho}(\theta)$ and $\hat{a}$ is not touched. In formulas,

$$\begin{aligned}\mathrm{tr}\{F[\hat{a}\exp(-\mathrm{i}\theta)]\hat{\rho}\} &= \mathrm{tr}\{F(\hat{U}^\dagger\hat{a}\hat{U})\hat{\rho}\}\\ &= \mathrm{tr}\{F(\hat{a})\hat{U}\hat{\rho}\hat{U}^\dagger\},\end{aligned} \tag{2.11}$$

as is easily verified by expanding F in powers of $\hat{a}$. We note that this idea of replacing a change in the observables by a change of the state is no different from the transition from a Heisenberg to a Schrödinger picture in quantum mechanics.

Finally, we introduce a pair of operators, $\hat{q}$ and $\hat{p}$, called the *quadratures*. They appear as the "real" and the "imaginary" part, respectively, of the "complex" amplitude $\hat{a}$ multiplied by $2^{1/2}$:

$$\hat{q} = 2^{-1/2}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = \mathrm{i}2^{-1/2}(\hat{a}^\dagger - \hat{a}) \tag{2.12}$$

so that

$$\hat{a} = 2^{-1/2}(\hat{q} + \mathrm{i}\hat{p}). \tag{2.13}$$

In optics $\hat{q}$ and $\hat{p}$ correspond to the in-phase and the out-of-phase component of the electric field amplitude of the spatial–temporal mode (with respect to a reference phase). It is easy to see from the basic bosonic commutation relation (2.3) that $\hat{q}$ and $\hat{p}$ are canonically conjugate observables,

$$[\hat{q}, \hat{p}] = \mathrm{i}. \tag{2.14}$$

(Note that $\hbar = 1$.) The quadratures $\hat{q}$ and $\hat{p}$ can be regarded as the position and the momentum of the electromagnetic oscillator. Of course, they do not appear in real space but in the phase space spanned by the complex vibrational amplitude $\hat{a}$ of the electromagnetic oscillator, and they have nothing to do with the position and the momentum of a photon (concepts that are problematic in any case). Nevertheless, the canonical commutation relation (2.14) entitles us to treat $\hat{q}$ and $\hat{p}$ as perfect examples of position- and momentumlike quantities. We will see later in this book that this analogy is one of the key points why quantum optics allows some fundamental Gedanken experiments of quantum physics to be carried out – not literally but certainly in the spirit of their inventors. We note that phase shifting rotates the quadratures

$$\hat{q}_\theta \equiv \hat{U}^\dagger(\theta)\hat{q}\hat{U}(\theta) = \hat{q}\cos\theta + \hat{p}\sin\theta \tag{2.15}$$

$$\hat{p}_\theta \equiv \hat{U}^\dagger(\theta)\hat{p}\hat{U}(\theta) = -\hat{q}\sin\theta + \hat{p}\cos\theta, \tag{2.16}$$

as is easily verified using definition (2.12) and the phase-shifting property (2.7) of the annihilation operator $\hat{a}$. We see that we can go from a position representation to a momentum representation via a phase shift θ of $\pi/2$. Finally, we express the photon-number operator $\hat{n}$ in terms of the quadratures $\hat{q}$ and $\hat{p}$ and obtain, using the canonical commutation relation (2.14),

$$\hat{H} \equiv \hat{n} + \frac{1}{2} = \frac{\hat{q}^2}{2} + \frac{\hat{p}^2}{2}. \tag{2.17}$$

The right-hand side of this equation stands for the energy of a harmonic oscillator with unity mass and frequency, that is, the photon number plus $1/2$ gives the energy of the electromagnetic oscillator. The additional $1/2$ is called the *vacuum energy* for a reason explained in Section 2.2.2.

2.2 Single-mode states

In this section we introduce several states of the electromagnetic oscillator that have a number of useful applications (in a purely mathematical or in a truly physical sense). We begin with the quadrature states, then turn to the Fock states, and consider finally coherent states as the most important realistic states of light. All states are introduced as eigenstates of prominent observables such as the quadratures, the photon number, and the annihilation operator.

2.2.1 Quadrature states

Let us call the eigenstates $|q\rangle$ and $|p\rangle$ of the quadratures $\hat{q}$ and $\hat{p}$ *quadrature states*, satisfying

$$\hat{q}|q\rangle = q|q\rangle, \qquad \hat{p}|p\rangle = p|p\rangle. \tag{2.18}$$

Because the quadratures obey the canonical commutation relation (2.14) their spectrum must be unbounded and continuous [58], as we would expect for position and momentum (see also Section 6.3). They are orthogonal

$$\langle q \,|\, q'\rangle = \delta(q - q'), \qquad \langle p \,|\, p'\rangle = \delta(p - p') \tag{2.19}$$

and complete

$$\int_{-\infty}^{+\infty} |q\rangle\langle q|\, dq = \int_{-\infty}^{+\infty} |p\rangle\langle p|\, dp = 1. \tag{2.20}$$

As is well known, position and momentum states are mutually related to each other by Fourier transformation

$$|q\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\mathrm{i}qp)|p\rangle\, dp \tag{2.21}$$

$$|p\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(+\mathrm{i}qp)|q\rangle\, dq. \tag{2.22}$$

However, the quadrature states are not truly normalizable, and so they cannot be generated experimentally (at least in a strict sense). Nevertheless, they will appear in many mathematical tricks. For instance, they are needed to introduce the *quadrature wave functions*

$$\psi(q) = \langle q \mid \psi\rangle, \qquad \tilde{\psi}(p) = \langle p \mid \psi\rangle. \tag{2.23}$$

In contrast to the quadrature states, the quadrature wave functions have a physical meaning. Their moduli squared account for the quadrature probability distributions $|\psi(q)|^2$ and $|\tilde{\psi}(p)|^2$ of the pure state $|\psi\rangle$, which can be precisely measured using homodyne detection, as will be considered in detail in Section 4.2.

2.2.2 Fock states

Let us introduce *Fock states*, or $|n\rangle$, as the eigenstates of the photon-number operator $\hat{n}$

$$\hat{n}|n\rangle = n|n\rangle. \tag{2.24}$$

Fock states are named after the Russian physicist V.A. Fock and are widely used in quantum field theory. As eigenstates of the number operator $\hat{n}$, Fock states have a perfectly fixed photon number. They possess appealing physical properties but are difficult to generate with present technology; see for instance Refs. [117] and [134] and the references cited therein.

Let us study the Fock states in some detail. First, we see that if $|n\rangle$ is an eigenstate of $\hat{n}$, then $\hat{a}|n\rangle$ must be an eigenstate as well, with the eigenvalue $n-1$. In fact,

$$\hat{n}\hat{a}|n\rangle = \hat{a}^\dagger\hat{a}^2|n\rangle = \left(\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}\right)|n\rangle = (n-1)\hat{a}|n\rangle. \tag{2.25}$$

In a similar way we easily show that $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{n}$ with the eigenvalue $n+1$. So we derive the fundamental relations

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{2.26}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.27}$$

The prefactors have been obtained using the fact that $\langle n|\hat{a}^\dagger\hat{a}|n\rangle$ must equal the eigenvalue n. Because of these relations, $\hat{a}$ is called the *annihilation operator* (it takes one photon out of $|n-1\rangle$) and $\hat{a}^\dagger$ is called the *creation operator*. The annihilation operator or the creation operator lowers or raises the photon number in integer steps. What would happen if we had a Fock state with noninteger eigenvalue n? A sufficiently large number of lowerings would certainly produce a Fock state with a photon number less than $-1/2$. On the other hand, we know from the relation (2.17) of $\hat{n}$ to the energy $\hat{H}$ that the average

$$\langle\hat{n}\rangle = \left\langle\frac{\hat{q}^2}{2}\right\rangle + \left\langle\frac{\hat{p}^2}{2}\right\rangle - \frac{1}{2} \geq -\frac{1}{2}. \tag{2.28}$$

This bound leads to a contradiction, because for eigenstates of $\hat{n}$ the average $\langle\hat{n}\rangle$ should equal the eigenvalue n. Consequently, no fractional photons exist, at least if the photon number is fixed precisely, that is, for photon-number eigenstates.

What happens if the photon number is integer – if we reach zero after lowering n in integer steps? Two options satisfy

$$\hat{a}^\dagger\hat{a}|0\rangle = 0. \tag{2.29}$$

One is to require that

$$\hat{a}|0\rangle = 0. \tag{2.30}$$

The other that

$$\hat{a}|0\rangle \neq 0, \quad \text{but} \quad \hat{a}^\dagger(\hat{a}|0\rangle) = 0. \tag{2.31}$$

Let us study the first option (2.30) first. Using the quadrature decomposition (2.13) of the annihilation operator and Schrödinger's famous formula $\hat{p} = -i\partial/\partial q$ in the q-representation, we obtain a differential equation for the wave function $\psi_0(q)$ of the state $|0\rangle$

$$\hat{a}\,\psi_0(q) = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right)\psi_0(q) = 0. \tag{2.32}$$

The solution of this equation is

$$\psi_0(q) = \pi^{-1/4}\exp\left(-\frac{q^2}{2}\right) \tag{2.33}$$

(normalized to yield $\int_{-\infty}^{+\infty}|\psi_0(q)|^2\,dq = 1$). In the momentum representation we obtain the same formula for $\tilde{\psi}_0(p)$

$$\tilde{\psi}_0(p) = \pi^{-1/4}\exp\left(-\frac{p^2}{2}\right). \tag{2.34}$$

In this way we have shown that a well-behaved state with precisely zero photons called the *vacuum state* exists. So even if the spatial–temporal mode is

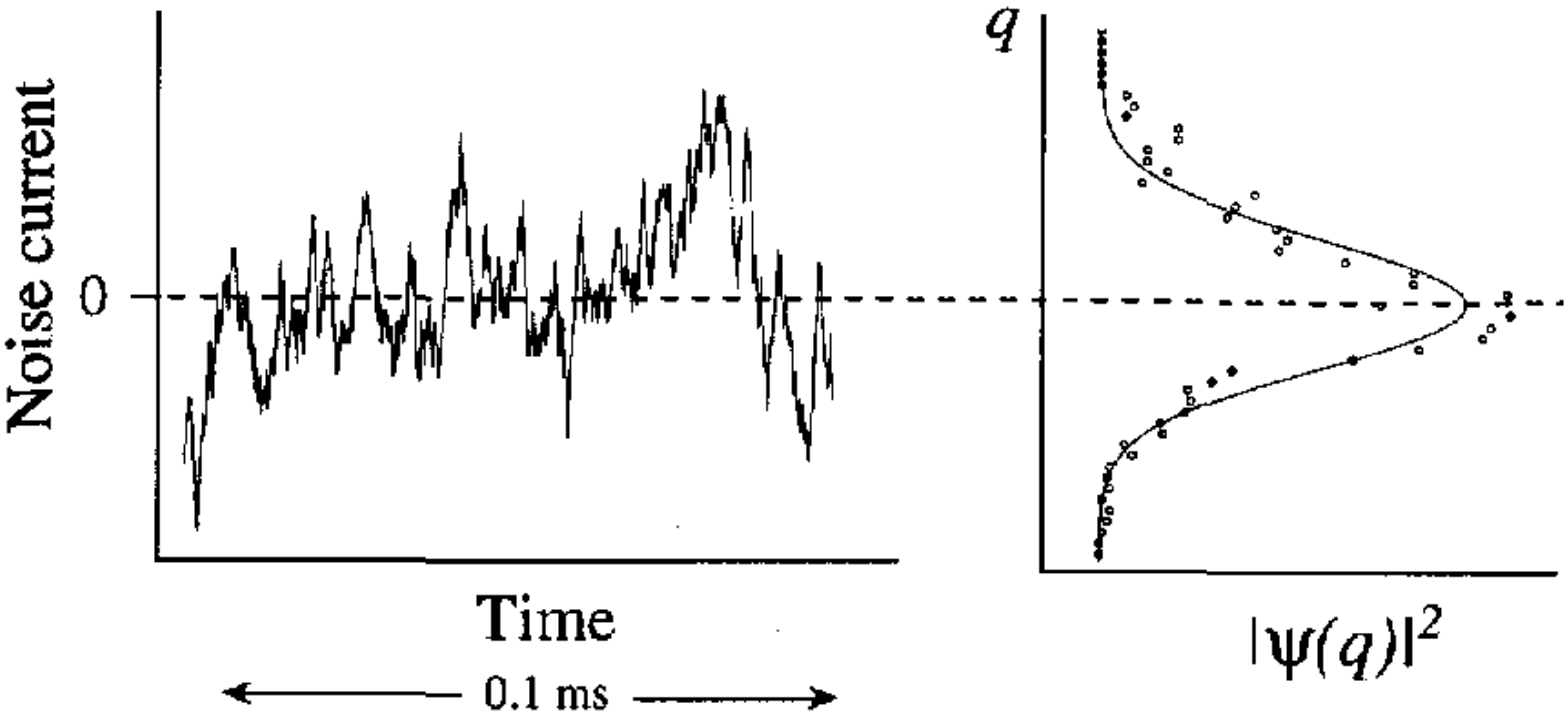

Fig. 2.1. Measurement of the vacuum noise. The position quadrature of an empty field was measured using balanced homodyne detection (see Section 4.2). Although the measurement time (0.1 milliseconds) is rather short, so that the number of samples is relatively low, the histogram of the noise current (dots) is approximately Gaussian and already follows the theoretical expectation (solid curve). [Courtesy of G. Breitenbach, University of Constance.]

completely empty, a physically meaningful state that might cause physical effects is still associated with this "emptiness."* Figure 2.1 shows a plot of the quadrature probability distribution $|\psi_0(q)|^2$ for a vacuum that has been measured using homodyne detection. (For an analysis of homodyne detection, see Section 4.2.) This curve illustrates beautifully that even in a complete vacuum the quadratures are still restlessly fluctuating. (This is the zero-point motion.) Of course, they must fluctuate; if both position and momentum quadratures were fixed, Heisenberg's uncertainty principle would be violated. The fluctuation energy of the vacuum state gives rise to the vacuum term 1/2 in the energy (2.17) of the electromagnetic oscillator.

Excited states are solutions of the relation (2.27) for an initial vacuum

$$|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}}|0\rangle. \tag{2.35}$$

We obtain a formula for their wave functions by expressing the relation (2.27) for $n = m - 1$ in the Schrödinger representation

$$\hat{a}^\dagger \psi_{m-1}(q) = \frac{1}{\sqrt{2}}\left(q - \frac{\partial}{\partial q}\right)\psi_{m-1}(q) = \sqrt{m}\,\psi_m(q). \tag{2.36}$$

This formula is satisfied by

$$\psi_n(q) = \frac{H_n(q)}{\sqrt{2^n n! \sqrt{\pi}}}\exp\left(-\frac{q^2}{2}\right). \tag{2.37}$$

*Throughout this book we always mean by "vacuum" simply "no light" and not an evacuated system.

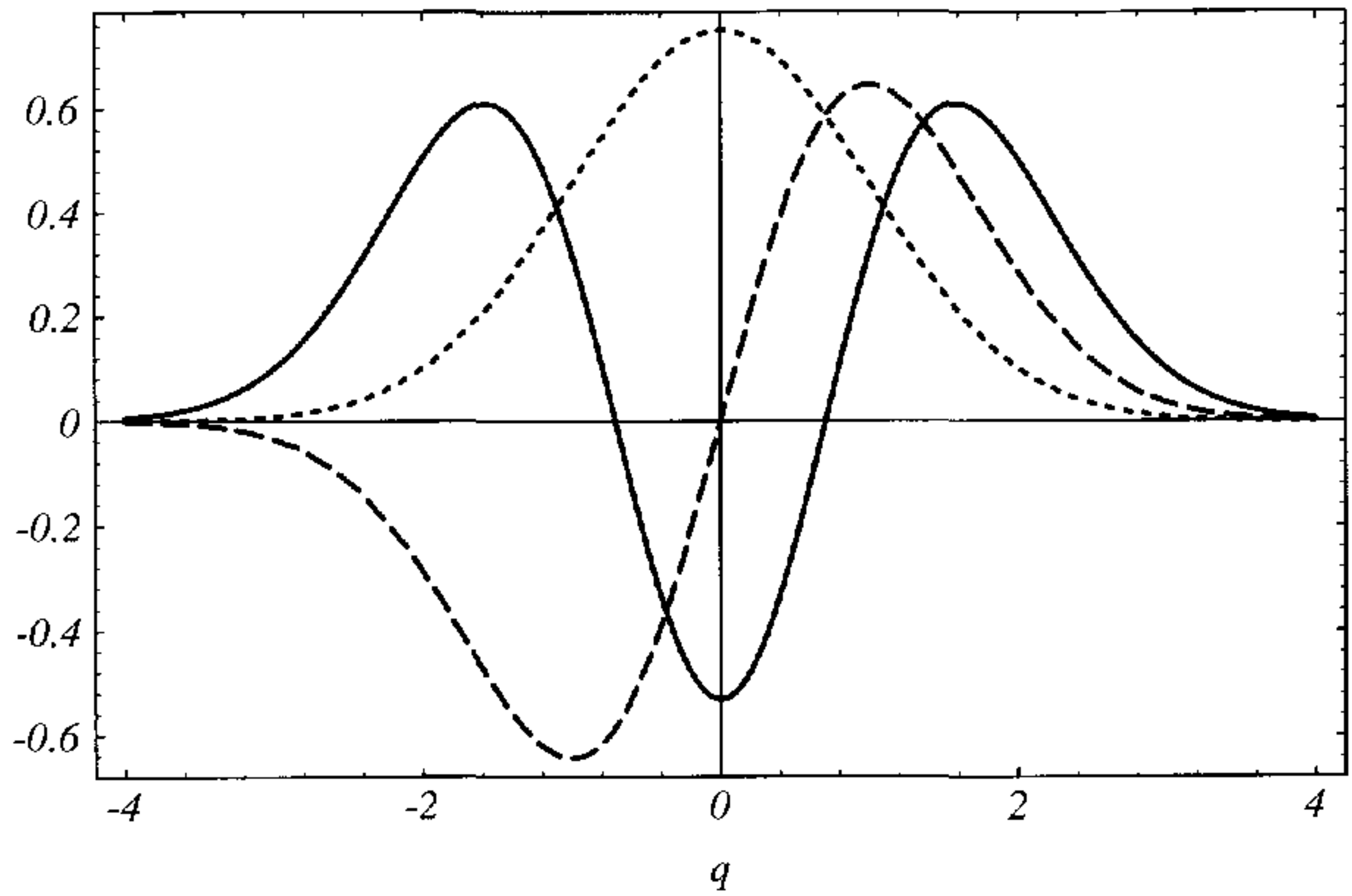

Fig. 2.2. Plot of the quadrature wave functions for some Fock states. Dotted line: vacuum ($n = 0$), dashed line: first excited state ($n = 1$), and solid line: second excited state ($n = 2$). The wave functions are even, $\psi_n(-q) = \psi_n(q)$, for even numbers n and odd, $\psi_n(-q) = -\psi_n(q)$, for odd numbers. They oscillate in the classically allowed region between the turning points of a classical harmonic oscillator with energy $n + 1/2$. Outside this region, that is, in the classically forbidden zone, the wave functions decay exponentially.

Here the H_n denote the Hermite polynomials, and we have used relation 10.13(14) of Ref. [89], Vol. II. Because we know that the Fock wave function for $n = 0$ is the vacuum wave function ψ_0 given by Eq. (2.33), we have found the ψ_n uniquely. Figure 2.2 shows plots of some Fock wave functions. They appear as standing Schrödinger waves for quadrature values ranging between the Bohr–Sommerfeld bands $-(2n+1)^{1/2}$ and $(2n+1)^{1/2}$. This behavior can be verified using the semiclassical theory for energy eigenstates described in Appendix 1. Consequently, the quadrature distributions, or squared wave functions, are broad. They illustrate that because the Fock states are particlelike, they have noisy quadrature amplitudes and exhibit few features of a classical, stable wave.

Let us return to the second possibility (2.31) for a vacuum state with a wave function $\varphi_0(q)$. It means that the function

$$\varphi_{-1}(q) \equiv \hat{a}\varphi_0(q) = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right)\varphi_0(q) \tag{2.38}$$

satisfies

$$\hat{a}^\dagger \varphi_{-1}(q) = \frac{1}{\sqrt{2}}\left(q - \frac{\partial}{\partial q}\right)\varphi_{-1}(q) = 0. \tag{2.39}$$

However, the solution of this equation

$$\varphi_{-1}(q) = c \exp\left(+\frac{q^2}{2}\right) \tag{2.40}$$

is not normalizable. Hence $\varphi_0(q)$, or the inhomogeneous solution of the differential equation (2.38), is not normalizable as well. It is called the *irregular wave function* of the vacuum state and is given by the expression

$$\varphi_0(q) = c\sqrt{\frac{\pi}{2}} \exp\left(-\frac{q^2}{2}\right) \mathrm{erfi}(q) \tag{2.41}$$

with erfi being the imaginary error function

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(t^2)\, dt. \tag{2.42}$$

Because the irregular vacuum is not normalizable, it must be rejected as a physically meaningful state. However, we will encounter later a useful mathematical application of irregular wave functions.

We have shown that a unique vacuum state exists for the electromagnetic oscillator and that all Fock states are given as excitations (2.35) of the vacuum. (Because the Schrödinger equation is of second order there are only two fundamental solutions – regular and irregular wave functions. The irregular ones are discarded as physical states.) So the Fock states must be complete,

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \tag{2.43}$$

that is, they span the whole Hilbert space of the electromagnetic oscillator. Additionally, Fock states are orthonormal,

$$\langle n \mid n' \rangle = \delta_{nn'}, \tag{2.44}$$

because they are eigenstates of the Hermitian operator $\hat{n}$. The Fock states form the most convenient and most frequently used orthonormal Hilbert-space basis in quantum optics, called the *Fock basis*.

2.2.3 Coherent states

We introduce *coherent states* as the eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.45}$$

Coherent states are also named *Glauber states* after the American physicist R.J. Glauber. They are called coherent states because light fields in these states are perfectly coherent in the sense of Ref. [187]. High-quality lasers generate such fields. As eigenstates of the annihilation operator $\hat{a}$, the coherent states have well-defined amplitudes $|\alpha|$ and phases $\arg\alpha$. (Because the annihilation operator $\hat{a}$ is not Hermitian, the eigenvalues of $\hat{a}$ are complex. They correspond to the complex wave amplitudes in classical optics.) Coherent states come as close as quantum mechanics allows to wavelike states of the electromagnetic oscillator. Because the wave aspects of light are commonly regarded as classical, coherent states are often called classical states. Furthermore, fields in statistical mixtures of coherent states (such as thermal fields) are classical as well, whereas any state that cannot be understood as an ensemble of coherent states is called *nonclassical.* The experimental generation and application of nonclassical light fields is one of the top issues of modern quantum optics. Despite much progress made, producing nonclassical states is still extremely challenging because they are easily destroyed (reduced to classical) by any kind of losses.

Let us return to coherent states and study their properties. First we note that vacuum is a coherent state as well, because it satisfies Eq. (2.45) for $\alpha = 0$, that is, vacuum is a zero-amplitude coherent state. Without much mathematical effort we see directly from the definition (2.45) that the mean energy of a coherent state is simply

$$\langle \hat{H} \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} + \frac{1}{2} | \alpha \rangle = |\alpha|^2 + \frac{1}{2}, \tag{2.46}$$

or the sum of the classical wave intensity $|\alpha|^2$ and the vacuum energy $1/2$. We also see easily from the definition (2.45) that a phase shift by the angle θ simply shifts the phase $\arg \alpha$ of the coherent-state amplitude

$$\hat{U}(\theta)|\alpha\rangle = |\alpha \exp(-\mathrm{i}\theta)\rangle. \tag{2.47}$$

This result is what we would expect for wavelike states.

To study coherent states more carefully, we introduce the *displacement operator*

$$\hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right). \tag{2.48}$$

Because $\mathrm{i}(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is Hermitian, $\hat{D}$ must be unitary. The displacement operator displaces the amplitude $\hat{a}$ by the complex number α

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha. \tag{2.49}$$

To prove this statement we imagine the displacement α as being decomposed into infinitesimal steps $\delta\alpha$ and we obtain in first order of $\delta\alpha$

$$\hat{D}^\dagger(\delta\alpha)\hat{a}\hat{D}(\delta\alpha) = \hat{a} + \left[\hat{a}, \hat{a}^\dagger\delta\alpha - \hat{a}\delta\alpha^*\right] = \hat{a} + \delta\alpha. \tag{2.50}$$

Because the total displacement operator $\hat{D}(\alpha)$ with $\alpha = \sum \delta\alpha$ is the product $\Pi \hat{D}(\delta\alpha)$ of the infinitesimal displacements $\hat{D}(\delta\alpha)$, we can apply the infinitesimal steps (2.50) as often as we need to show that $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)$ equals indeed $\hat{a} + \sum \delta\alpha$, which proves Eq. (2.49). (Readers who are familiar with Lie groups notice easily that we have used the properties of the Lie algebra to study the group elements. Quite generally, coherent states are intimately linked to Lie groups; see Ref. [216]). What has the displacement operator to do with coherent states? Let us apply a "negative" displacement to $|\alpha\rangle$. We see from the basic property (2.49) of the displacement operator that

$$\begin{aligned} \hat{a}\hat{D}(-\alpha)|\alpha\rangle &= \hat{D}(-\alpha)\hat{D}^\dagger(-\alpha)\hat{a}\hat{D}(-\alpha)|\alpha\rangle \\ &= \hat{D}(-\alpha)(\hat{a}-\alpha)|\alpha\rangle, \end{aligned} \tag{2.51}$$

which must equal zero because of the definition (2.45) of coherent states. This implies that $\hat{D}(-\alpha)|\alpha\rangle$ is *the* vacuum state $|0\rangle$. Consequently, coherent states $|\alpha\rangle$ are *displaced vacuums*

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{2.52}$$

Of course, this does not mean that coherent states are physically similar to vacuum states. They have only some quantum-noise properties in common. (Displacing the vacuum may appear as rather inappropriate description of generating high-quality laser light.)

To study the relation between coherent states and the vacuum in more detail, we calculate the quadrature wave functions $\psi_\alpha(q)$ and $\tilde{\psi}_\alpha(p)$. We decompose the complex amplitude α into real and imaginary parts

$$\alpha = 2^{-1/2}(q_0 + \mathrm{i}p_0) \tag{2.53}$$

and represent the displacement operator in terms of the quadratures $\hat{q}$ and $\hat{p}$

$$\hat{D} = \exp(\mathrm{i}p_0\hat{q} - \mathrm{i}q_0\hat{p}). \tag{2.54}$$

We take advantage of the *Baker–Hausdorff formula*

$$\begin{aligned} \exp(\hat{F}+\hat{G}) &= \exp\left(-\frac{1}{2}[\hat{F},\hat{G}]\right)\exp(\hat{F})\exp(\hat{G}) \\ &= \exp\left(+\frac{1}{2}[\hat{F},\hat{G}]\right)\exp(\hat{G})\exp(\hat{F}) \end{aligned} \tag{2.55}$$

for any two operators $\hat{F}$ and $\hat{G}$ such that the commutator $[\hat{F},\hat{G}]$ commutes with both of them. This fundamental operator relation is proven for instance in Ref. [100]. Here we use the Baker–Hausdorff formula to split $\hat{D}$ into three parts

$$\hat{D}(\alpha) = \exp\left(-\frac{\mathrm{i}p_0q_0}{2}\right)\exp(\mathrm{i}p_0\hat{q})\exp(-\mathrm{i}q_0\hat{p}) \tag{2.56}$$

$$= \exp\left(+\frac{\mathrm{i}p_0q_0}{2}\right)\exp(-\mathrm{i}q_0\hat{p})\exp(\mathrm{i}p_0\hat{q}). \tag{2.57}$$

In the position representation the momentum operator $\hat{p}$ equals $-\mathrm{i}\partial/\partial q$ and the exponential $\exp(-q_0\partial/\partial q)$ is a translation operator

$$\exp\left(-q_0\frac{\partial}{\partial q}\right)\psi(q) = \psi(q-q_0), \tag{2.58}$$

as it is easily verified by differentiating both sides with respect to q_0. In this way we realize that the displacement operator acts in three steps on position wave functions: First it displaces the wave function; then it multiplies it with $\exp(\mathrm{i}p_0\hat{q})$, that is, with $\exp(\mathrm{i}p_0 q)$ in the position representation; and finally, the displacement operator attaches the phase factor $\exp(-\mathrm{i}p_0q_0/2)$ to the wave function. Because coherent states are displaced vacuums the position wave function is simply

$$\psi_\alpha(q) = \psi_0(q-q_0)\exp\left(+\mathrm{i}p_0 q - \frac{\mathrm{i}p_0q_0}{2}\right) \tag{2.59}$$

$$= \pi^{-1/4}\exp\left[-\frac{(q-q_0)^2}{2} + \mathrm{i}p_0 q - \frac{\mathrm{i}p_0q_0}{2}\right], \tag{2.60}$$

with ψ_0 being the vacuum wave function. In a similar way we obtain the momentum wave function

$$\tilde{\psi}_\alpha(p) = \pi^{-1/4}\exp\left[-\frac{(p-p_0)^2}{2} - \mathrm{i}q_0 p + \frac{\mathrm{i}p_0q_0}{2}\right]. \tag{2.61}$$

The Eqs. (2.60) and (2.61) show that the quadrature probability distributions $|\psi_\alpha(q)|^2$ and $|\tilde{\psi}_\alpha(p)|^2$ of coherent states are Gaussian with the same width as the Gaussian curve for vacuum. They are shifted only by the real amplitudes q_0 and p_0, and we would obtain a completely similar picture for the coherent states as for the vacuum in Fig. 2.1 (see Fig. 3.2). In this sense coherent states are similar to the vacuum. Only the vacuum fluctuations contaminate the quadrature amplitudes, illustrating that coherent states are wavelike – they have just as much quadrature noise as is unavoidable. This reason is why high-quality laser light is a wonderful tool for experimentation.

So much for the wave features of coherent states; let us now study the particle aspects. For this purpose we seek the Fock representation of $\hat{D}(\alpha)|0\rangle$. We express $\hat{D}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, Eq. (2.48), and use the Baker–Hausdorff formula (2.55) again to split the displacement operator into three parts

$$\hat{D} = \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp\left(\alpha\hat{a}^\dagger\right)\exp(-\alpha^*\hat{a}). \tag{2.62}$$

Because the annihilation operator annihilates the vacuum, all powers of $\hat{a}$ contained in the Taylor series expansion of the exponential $\exp(-\alpha^*\hat{a})$ give zero when applied to $|0\rangle$, with the only exception being the zeroth order term 1. So the exponential $\exp(-\alpha^*\hat{a})$ does not affect the vacuum state at all. We expand the other exponential $\exp(\alpha\hat{a}^\dagger)$ in the splitting (2.62) in the Taylor series and

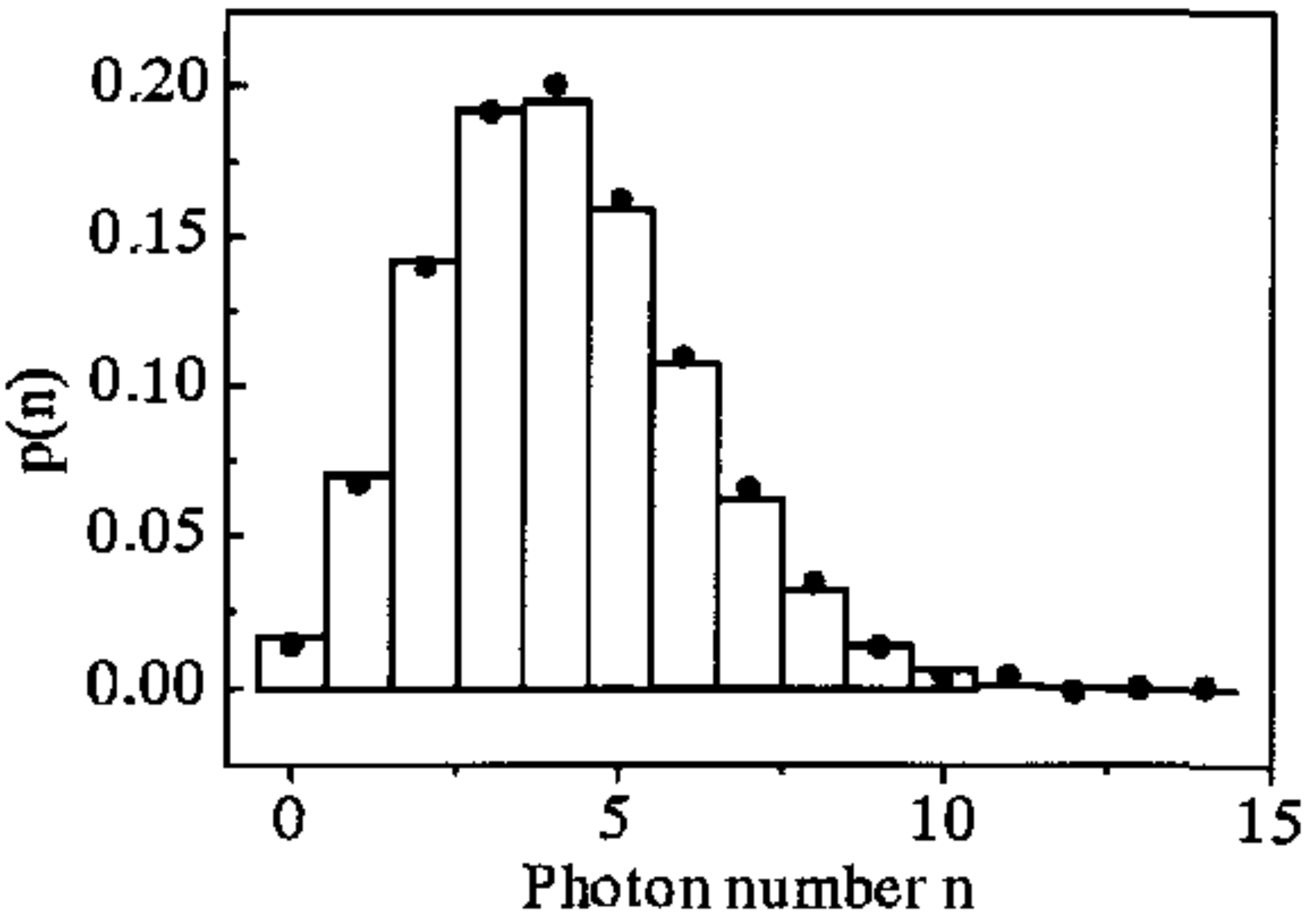

Fig. 2.3. Photon-number distribution of light in a coherent state. The distribution was obtained from experimental homodyne data via the method described in Section 5.2. We see that the reconstructed histogram (dots) is approximately Poissonian (bar chart). [Courtesy of G. Breitenbach, University of Constance.]

use formula (2.35) for the Fock states $|n\rangle$ to obtain

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.63}$$

The Fock representation (2.63) shows that a coherent state has Poissonian photon statistics

$$p_n = |\langle n\,|\,\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2). \tag{2.64}$$

Counting the photons of a coherent state means making repeated measurements on a statistical ensemble of identically prepared fields. Each time, n photons are obtained with Poissonian probability p_n, and on average we get as many photons as quantified by the intensity $|\alpha|^2$. Classical particles obey the same statistical law when they are taken at random from a pool with an average of $|\alpha|^2$ each time. We may say that when the photons of a coherent state are counted they behave like randomly distributed classical particles. This classical randomness seems not too surprising because coherent states are wavelike.

Let us finally derive some formal properties of coherent states that turn out to be quite useful. First, we note that coherent states are not exactly orthogonal to each other because they are not eigenstates of a Hermitian operator. Instead, they are approximately orthogonal when their amplitudes differ sufficiently. In fact, we obtain from the Fock representation (2.63)

$$\begin{aligned}\langle\alpha'\,|\,\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2}-\frac{|\alpha'|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\alpha'^*\alpha)^n}{n!}\\ &= \exp\left(-\frac{|\alpha|^2}{2}-\frac{|\alpha'|^2}{2}+\alpha'^*\alpha\right)\end{aligned} \tag{2.65}$$

and consequently

$$|\langle\alpha'\,|\,\alpha\rangle|^2 = \exp(-|\alpha-\alpha'|^2). \tag{2.66}$$

The Gaussian in Eq. (2.66) goes rapidly to zero when the amplitudes α and α' differ significantly more than the quadrature-noise level of the vacuum. We also note that coherent states are complete,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\alpha\rangle\langle\alpha|\frac{dq_0\,dp_0}{2\pi} = 1. \tag{2.67}$$

That is, we may express physical quantities in a coherent-state basis. (Coherent states are even overcomplete because fewer than all of them form a basis already. See Refs. [9] and [50]. This property is, by the way, a side effect of their lack of strict orthogonality.) The proof of the completeness relation (2.67) is a matter of substituting the Fock representation for $|\alpha\rangle$ and performing the necessary integrations using polar coordinates in the complex plane.

2.3 Uncertainty and squeezing

In the preceding section we introduced Fock states and coherent states as some physically meaningful states of the electromagnetic oscillator. We have seen that the quadrature amplitudes of these states fluctuate according to certain probability distributions. Coherent states are distinguished for having only as much statistical uncertainty in their quadrature amplitudes as the vacuum. Is this the quantum-mechanical optimum, or, to put it differently, what are the *minimum-uncertainty states*? Pauli settled this question in a brilliant few-line proof published in his *Handbuch der Physik* article [212]. Let us first denote the average complex amplitude of possible candidate states $|\psi\rangle$ by α

$$\langle\psi|\hat{a}|\psi\rangle = \alpha = 2^{-1/2}(q_0 + \mathrm{i}p_0). \tag{2.68}$$

We remove the amplitude α from consideration by applying the displacement operator to obtain a new state, $|\varphi\rangle$,

$$|\varphi\rangle = \hat{D}(-\alpha)|\psi\rangle \tag{2.69}$$

that contains the same amount of quadrature noise as the state $|\psi\rangle$. The variances of q and p are given by

$$\Delta^2 q = \langle\psi|(\hat{q}-q_0)^2|\psi\rangle = \langle\varphi|\hat{q}^2|\varphi\rangle \tag{2.70}$$

and

$$\Delta^2 p = \langle\psi|(\hat{p}-p_0)^2|\psi\rangle = \langle\varphi|\hat{p}^2|\varphi\rangle. \tag{2.71}$$

Now we use Pauli's argument [212] and state that for the position wave function $\varphi(q)$ of $|\varphi\rangle$ the quantity

$$\delta \equiv \left| \frac{q}{2\Delta^2 q}\varphi + \frac{\partial\varphi}{\partial q} \right|^2 \tag{2.72}$$

must necessarily be greater than or at least equal to zero. On the other hand,

$$\begin{aligned}
\delta &= \frac{1}{4}\left(\frac{q}{\Delta^2 q}\right)^2 \varphi^*\varphi + \frac{q}{2\Delta^2 q}\left(\varphi\frac{\partial\varphi^*}{\partial q} + \varphi^*\frac{\partial\varphi}{\partial q}\right) + \frac{\partial\varphi^*}{\partial q}\frac{\partial\varphi}{\partial q} \\
&= \frac{1}{4}\left(\frac{q}{\Delta^2 q}\right)^2 \varphi^*\varphi + \frac{1}{2}\frac{\partial}{\partial q}\left(\frac{q}{\Delta^2 q}\varphi^*\varphi\right) - \frac{1}{2}\frac{\varphi^*\varphi}{\Delta^2 q} + \frac{\partial\varphi^*}{\partial q}\frac{\partial\varphi}{\partial q} \\
&= \frac{1}{4(\Delta^2 q)^2}(q^2 - 2\Delta^2 q)\varphi^*\varphi + \frac{\partial\varphi^*}{\partial q}\frac{\partial\varphi}{\partial q} + \frac{1}{2}\frac{\partial}{\partial q}\left(\frac{q}{\Delta^2 q}\varphi^*\varphi\right).
\end{aligned} \tag{2.73}$$

We integrate the last line, drop the total differential $\partial(\varphi^*\varphi q/\Delta^2 q)/\partial q$ and use $\hat{p} = -\mathrm{i}\partial/\partial q$ for the $(\partial\varphi^*/\partial q)\cdot(\partial\varphi/\partial q)$ term to see that

$$\int_{-\infty}^{+\infty} \delta\, dq = -\frac{1}{4\Delta^2 q} + \Delta^2 p \geq 0 \tag{2.74}$$

which implies for $\Delta q = \sqrt{\Delta^2 q}$ and $\Delta p = \sqrt{\Delta^2 p}$

$$\Delta q \Delta p \geq \frac{1}{2}. \tag{2.75}$$

This formula is nothing else than Heisenberg's uncertainty relation (with $\hbar$ being scaled to unity). Because Pauli used the integration of δ in his derivation, he produced essentially a local version of Heisenberg's uncertainty principle. The great advantage of this mathematical trick is that it reveals the minimum-uncertainty states in the twinkling of an eye. In fact, because $\delta \geq 0$, the equality sign in relation (2.75) holds only if

$$\frac{1}{2}\frac{q}{\Delta^2 q}\varphi + \frac{\partial\varphi}{\partial q} = 0. \tag{2.76}$$

The normalized solution of this differential equation is

$$\varphi(q) = \left(\pi 2\Delta^2 q\right)^{-1/4} \exp\left[-\frac{q^2}{4\Delta^2 q}\right]. \tag{2.77}$$

So apart from a displacement, the minimum-uncertainty states have Gaussian wave functions like coherent states. However, the variance $\Delta^2 q$ should not necessarily equal $1/2$, as is the case for coherent states. In other words, both variances $\Delta^2 q$ and $\Delta^2 p$ are not required to be equal to minimize Heisenberg's uncertainty relation (2.75). The statistical uncertainty of the position quadrature q may be *squeezed* below the vacuum level $1/2$ at the cost, however, of enhancing the uncertainty in the canonically conjugate quadrature p and vice versa.

Let us study this squeezing effect more carefully. We parameterize the deviation of the variances from their vacuum values by a real number ζ called the *squeezing parameter*

$$\Delta^2 q = \frac{1}{2}\mathrm{e}^{-2\zeta}, \qquad \Delta^2 p = \frac{1}{2}\mathrm{e}^{+2\zeta}. \tag{2.78}$$

Obviously, the product of Δq and Δp equals the minimal value $1/2$. How can we squeeze the vacuum? Mathematically, we could just scale the position wave function ψ_0 for the vacuum

$$\varphi(q) = \mathrm{e}^{\zeta/2}\psi_0(\mathrm{e}^{\zeta} q). \tag{2.79}$$

The prefactor $\mathrm{e}^{\zeta/2}$ in this scaling serves for maintaining the normalization of $\varphi(q)$. The momentum wave function $\tilde{\varphi}(p)$ is the Fourier-transformed position wave function. Consequently,

$$\tilde{\varphi}(p) = \mathrm{e}^{-\zeta/2}\tilde{\psi}_0(\mathrm{e}^{-\zeta} p). \tag{2.80}$$

This formula implies that the momentum wave function is antisqueezed when the position wave function is squeezed and vice versa. We differentiate φ in Eq. (2.79) with respect to the squeezing parameter ζ and obtain

$$\frac{\partial\varphi}{\partial\zeta} = \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right)\varphi = \frac{1}{2}(\mathrm{i}\hat{q}\hat{p} + \mathrm{i}\hat{p}\hat{q})\varphi. \tag{2.81}$$

Since $\mathrm{i}\hat{q}\hat{p} + \mathrm{i}\hat{p}\hat{q}$ equals $\hat{a}^2 - \hat{a}^{\dagger 2}$ we can express the formal solution of this differential equation in terms of the unitary *squeezing operator*

$$\hat{S} \equiv \exp\left[\frac{\zeta}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)\right] \tag{2.82}$$

and obtain for the *squeezed-vacuum state*

$$|\varphi\rangle = \hat{S}(\zeta)|0\rangle. \tag{2.83}$$

According to Pauli's proof, all minimum-uncertainty states are displaced Gaussian states, that is, they have displaced rescaled vacuum wave functions. Consequently, all minimum-uncertainty states are *displaced squeezed vacuums*

$$|\psi\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle \tag{2.84}$$

having a position wave function of

$$\psi(q) = \pi^{-1/4}\mathrm{e}^{\zeta/2}\exp\left[-\mathrm{e}^{2\zeta}\frac{(q-q_0)^2}{2} + \mathrm{i}pq - \frac{\mathrm{i}p_0 q_0}{2}\right]. \tag{2.85}$$

In this way we have found not only a convenient mathematical notation for the squeezed states but also one possible physical process for generating squeezed light experimentally. We interpret simply the squeezing operator $\hat{S}(\zeta)$ as an evolution operator describing the result of the interaction

$$\hat{H}_{int} \propto \left(\hat{a}^2 - \hat{a}^{\dagger 2}\right). \tag{2.86}$$

The squeezing parameter ζ contains the product of the coupling strength and the interaction time. Processes described by nonlinear Hamiltonians such as $\hat{H}_{int}$ in Eq. (2.86) belong to the branch of *nonlinear optics*. In particular, the squeezing interaction (2.86) is realized by the degenerate parametric amplification of the spatial–temporal mode. A crystal such as KTP (potassium titanyl phosphate) is pumped by another laser beam at twice the frequency of the spatial–temporal mode of interest. The pump amplifies the signal parametrically much as a swing is amplified by changing the effective length at twice the frequency of the swing. A classical swing relies on tiny initial fluctuations that are in-phase with respect to the parametric pump. A quantum swing like the degenerate parametric amplifier experiences at least the vacuum fluctuations from the very beginning. Vacuum fluctuations that are in-phase with respect to the pump are amplified, whereas out-of-phase fluctuations get deamplified or, in other words, squeezed.

A squeezed vacuum requires a pump for generation, and, hence, when produced it carries energy. To quantify the amount of squeezing energy we note that the squeezing operator changes the quadratures

$$\hat{S}^{\dagger}(\zeta)\hat{q}\hat{S}(\zeta) = \hat{q}e^{-\zeta} \tag{2.87}$$

$$\hat{S}^{\dagger}(\zeta)\hat{p}\hat{S}(\zeta) = \hat{p}e^{+\zeta} \tag{2.88}$$

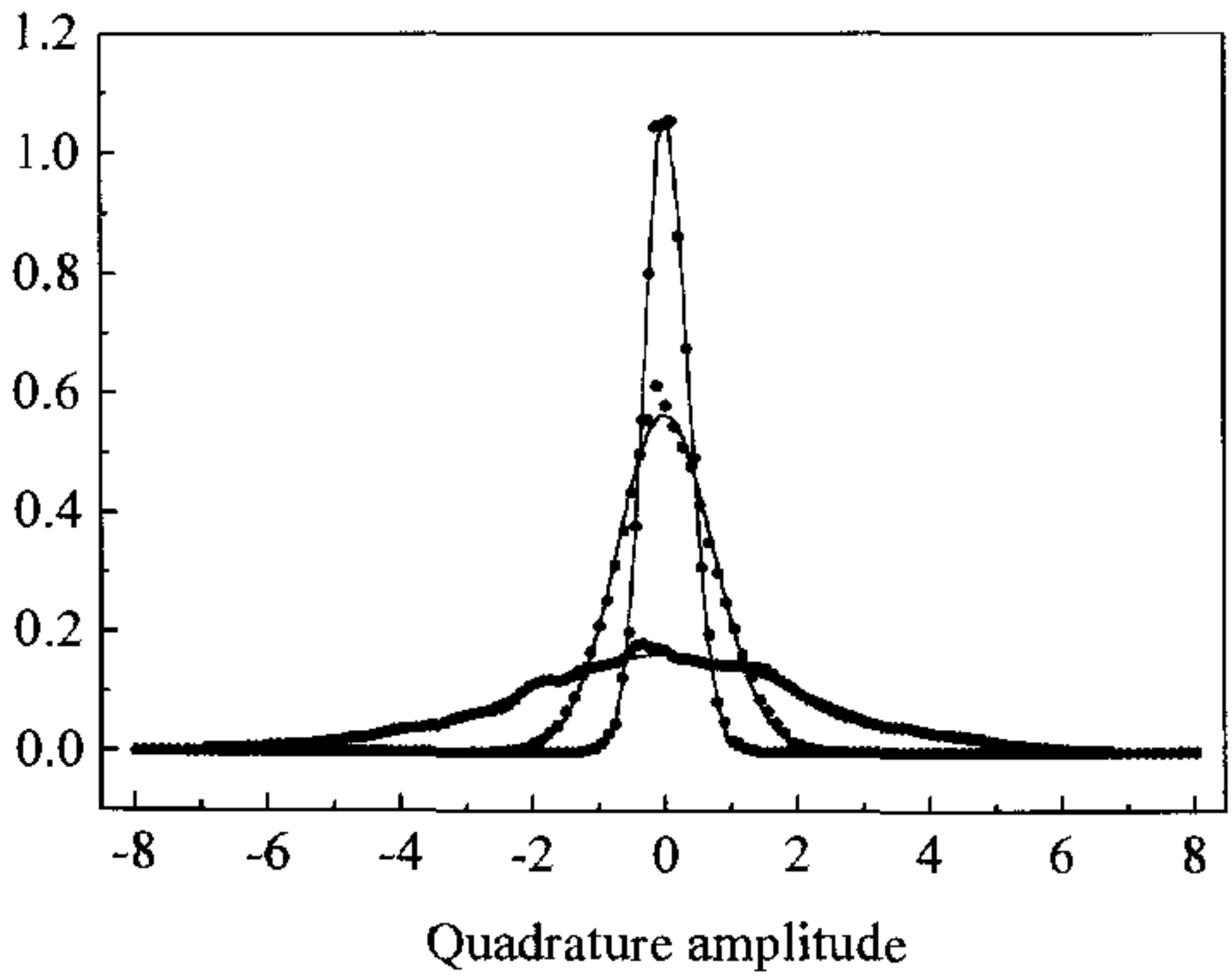

Fig. 2.4. Quadrature distributions of a squeezed vacuum [41]. Dots: measured values, lines: theoretical predictions. The squeezed and the antisqueezed components are depicted in comparison with the quadrature distribution of a vacuum (central plot).

because it scales the eigenfunctions of $\hat{q}$ and $\hat{p}$ accordingly. Substituting for $\hat{a}$ the quadrature decomposition (2.13) we see that

$$\hat{S}^\dagger(\zeta)\hat{a}\hat{S}(\zeta) = \hat{a}\cosh\zeta - \hat{a}^\dagger \sinh\zeta. \tag{2.89}$$

We use this formula to express the energy (2.17) of a squeezed state (2.84) and obtain

$$\langle\psi|\hat{H}|\psi\rangle = |\alpha|^2 + \frac{1}{2} + \sinh^2\zeta. \tag{2.90}$$

Three terms contribute to the energy. The first accounts for the coherent energy given by the modulus squared of the coherent amplitude α, the second is the vacuum energy $1/2$, and the last quantifies the fluctuation energy of squeezed states. Originally, the contribution to this squeezing energy comes from the pump used to generate the squeezed light. It is stored in the enhanced fluctuations of the antisqueezed component. Because both the squeezed and the antisqueezed quadratures contribute to $\hat{H} = (\hat{q}^2 + \hat{p}^2)/2$, even a squeezed vacuum carries energy.

Let us calculate the energy distribution, that is, the photon-number statistics of a squeezed vacuum

$$p_n = |\langle n|\hat{S}(\zeta)|0\rangle|^2. \tag{2.91}$$

We express the scalar product in the position representation

$$\langle n|\hat{S}(\zeta)|0\rangle = \int_{-\infty}^{+\infty} \psi_n(q) e^{\zeta/2} \psi_0(e^\zeta q)\, dq. \tag{2.92}$$

A squeezed vacuum as well as the vacuum state is perfectly symmetric if we flip the sign of the quadrature amplitude q. [It has an even wave function $\psi(q) = \psi(-q)$.] The wave functions $\psi_n(q)$ for the Fock states are even for even photon numbers and odd if n is odd. Consequently, the integral (2.92) vanishes for odd photon numbers and we obtain

$$p_{2m+1} = 0 \quad (m = 0, 1, 2, \ldots). \tag{2.93}$$

A squeezed vacuum contains only photon pairs. We may see this fact as a simple consequence of the mirror symmetry of squeezing. A deeper physical reason for this remarkable property is that a squeezed vacuum may be generated in a parametric process described by the quadratic Hamiltonian (2.86). Loosely speaking, photons are created in pairs: each pump photon is converted into two signal photons of half the pump frequency. The probability for finding a photon pair is

$$p_{2m} = \binom{2m}{m} \frac{1}{\cosh\zeta} \left(\frac{1}{2}\tanh\zeta\right)^{2m} \quad (m = 0, 1, 2, \ldots). \tag{2.94}$$

Here Eq. 2.20.3.3. of Ref. [225], Vol. II, has been used to perform the necessary integration (2.92). A simple explanation exists for formula (2.94) in terms of the statistics of classical particles much like the explanation for the Poissonian photon distribution of coherent states. Formula (2.94) appears like a probability distribution of independently produced particle pairs. Photons are generated independently from one another with a probability proportional to $1/2 \tanh \zeta$, that is, proportional to the half of the squeezing parameter ζ for weak pumping. For stronger pumping the generation process becomes saturated. We observe pairs of $2m$ independently produced photons. All photons appear as distinguishable classical particles, but the detector cannot discriminate between them. It detects any m pairs of $2m$ particles, giving rise to a statistical enhancement described by the binomial coefficient in formula (2.94). Note that although this explanation is consistent with the pair statistics (2.94), it loses its meaning when the wave features of light (for instance, interference effects) become important. Figure 2.5 shows an experimental plot of the photon statistics of a squeezed vacuum measured using a method described in detail in Section 5.2.2. We see clearly near-zero probability for odd photon numbers and a decreasing probability for an increasing number of pairs. (The small nonzero probability for odd numbers is caused by detection inefficiencies; see Section 4.1.4.) Because higher-number pairs are produced by higher-order processes, they are less likely to be observed than lower-number pairs. Vacuum has always the lion's share in the photon-number distribution of a squeezed vacuum.

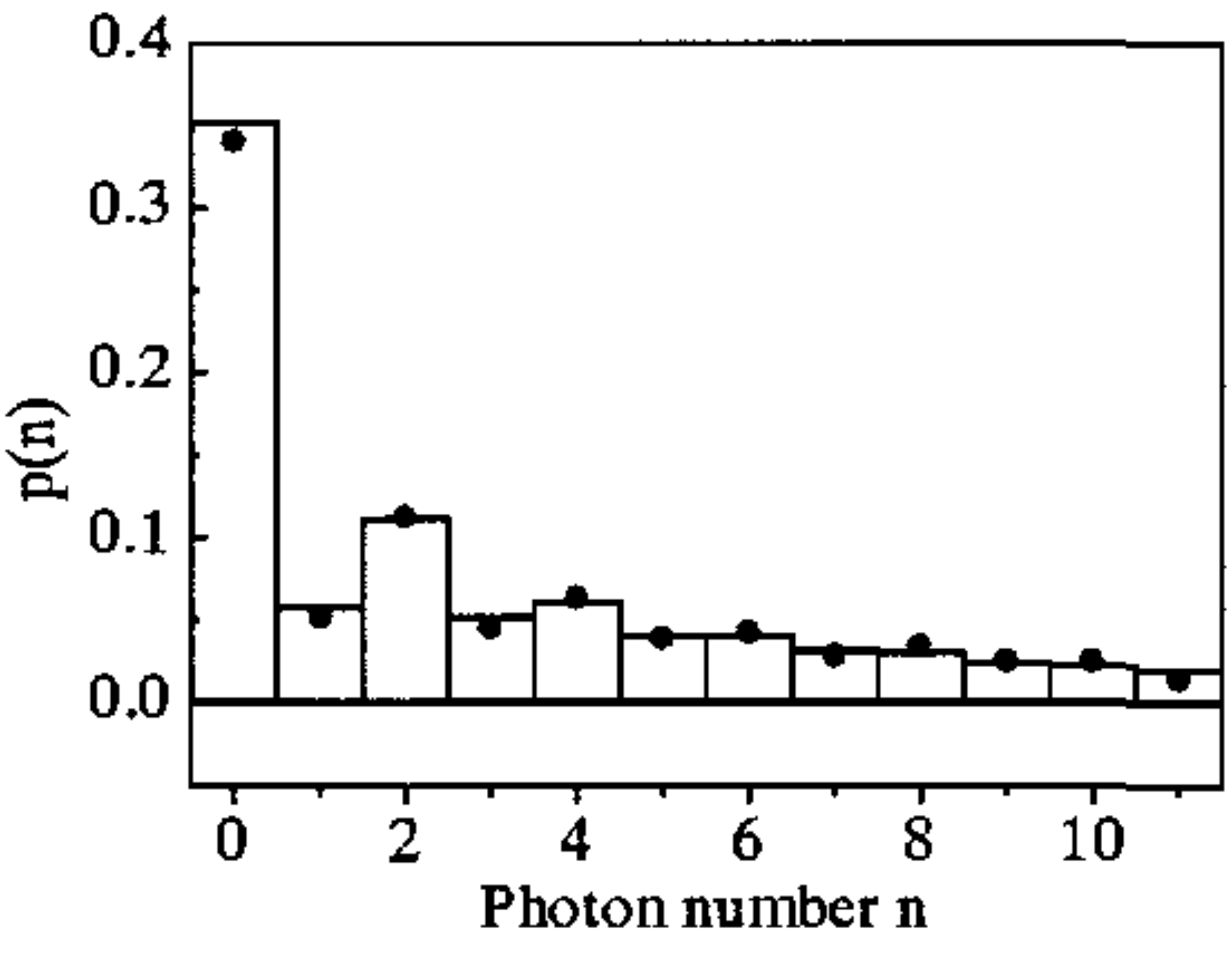

Fig. 2.5. Photon-number distribution of a squeezed vacuum. Photons are produced in pairs. However, detection inefficiencies (see Section 5.3.1) break the pairs so that we observe also odd photon numbers with nonzero probability. The histogram (dots) was reconstructed from measured quadrature distributions via the method described in Section 5.2. The bar chart shows a fit with the theoretical prediction based on Eqs. (2.93) and (2.94) and formula (4.59), taking detection losses into account. [Courtesy of G. Breitenbach, University of Constance.] See also Ref. [240].

Finally, we note that squeezed states are nonclassical states. Because they are pure states and different from coherent states, they cannot be described in terms of statistical mixtures of coherent states. Apart from this trivial formal statement, the quadrature noise reduction below the vacuum level and the photon pairing of squeezed vacuum illustrate beautifully that these states have indeed distinguished quantum properties.

2.4 Further reading

A more complete and detailed introduction to the quantum theory of light is given in most textbooks on quantum optics. For instance, the reader is referred to the books by H.J. Carmichael [55]; C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg [59, 60]; C.W. Gardiner [100]; H. Haken [108]; R. Loudon [178]; W.H. Louisell [181]; P. Meystre and M. Sargent [109]; H.M. Nussenzveig [202]; J. Peřina [217]; W.P. Schleich, D.S. Krähmer, and E. Mayr [245]; M.O. Scully and M.S. Zubairy [251]; D.F. Walls and G.J. Milburn [284]; and W. Vogel and D.-G. Welsch [281] and to the monumental monograph by L. Mandel and E. Wolf [187]. Various aspects of generalized coherent states are considered in the collection [138] of reprinted papers edited by J.R. Klauder and B.-S. Skagerstam. See also the book by A.M. Perelomov [216]. Squeezed states are reviewed by R. Loudon and P.L. Knight [179].